Drag reduction by compressible bubbles

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Drag reduction by bubbles in stationary turbulent flows is sensitive to the compressibility of the bubbles. Without this dynamical effect the bubbles only renormalize the fluid density and viscosity, an effect that by itself can only lead to a small percentage of drag reduction. We show in this paper that the dynamics of bubbles and their effect on the compressibility of the mixture can lead to a much higher drag reduction.

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I. INTRODUCTION

Drag reduction in turbulent flows is a subject of technological importance and of significant basic interest. As is well known, drag reduction can be achieved using a number of additives, including flexible polymers, rodlike polymers and fibers, surfactants, and bubbles [1]. While the subject of drag reduction by polymers had seen rapid theoretical progress in the last few years [2–7] the understanding of drag reduction by bubbles lags behind. For practical applications in the shipping industry the use of polymers is out of the question for economic and environmental reasons, but air bubbles are potentially very attractive.

The theory of drag reduction by small concentrations of minute bubbles is relatively straightforward, since under such conditions the bubbles only renormalize the density and the viscosity of the fluid, and a one-fluid model suffices to describe the dynamics [8]. The fluid remains incompressible, and the equations of motion are basically the same as for a Newtonian fluid with renormalized properties. The amount of drag reduction under such conditions is however, limited. But when the bubbles increase in size, the one-fluid model loses its validity since the bubbles become dynamical in the sense that they are no longer Lagrangian particles, their velocity is no longer the fluid velocity at their center, and they begin to fluctuate under the influence of local pressure variations. The fluctuations of the bubbles are of two types: (i) the bubbles are no longer spherical, distorting their shape according to the pressure variations, and (ii) the bubbles can change their volume (keeping their spherical shape) due to the compressibility of the gas inside the bubble. The first effect was studied numerically using the "front tracking" algorithm in Refs. [9,10]. However, the results indicate either a drag enhancement, or a limited and transient drag reduction. This leads one to study the possibility of explaining bubbly drag reduction by bubble compressibility. Indeed, a theoretical model proposed by Legner [11] successfully explained the bubbly drag reduction by modifying the turbulent viscosity in the bubbly flow by the bulk viscosity of the bubbles. While the bulk viscosity is important only when the bubbles are compressible, it is important and interesting to see how and why it affects the characteristics of the flow. The aim of this paper is to study the drag reduction by bubbles when bubble compressibility is dominant. Finally we compare our finding with the results in Ref. [11], showing that a nonphysical aspect of that theory is removed, while a good agreement with experiment is retained.

In our thinking we were influenced by two main findings, one experimental and the other simulational. The experiment [12] established the importance of bubble dynamics in effecting drag reduction. The same turbulent flow was set up once in the presence of bubbles and once in the presence of glass spheres whose density was smaller than that of the ambient fluid. While bubbles effected drag reduction for sufficiently high Reynolds number, the glass spheres enhanced the drag. In the simulation [13] it was demonstrated that the drag reduction by the bubbles is connected in an intimate way to the effective compressibility of the mixture. (The fluid by itself was taken as incompressible in the simulation.) These two observations, in addition to the experiments [14] will be at the back of our mind in developing the theory, with the final elucidation of all these observations in the last sections of this paper.

In Sec. II we present the average (field) equations for fluids laden with bubbles. This theory follows verbatim earlier work [15–17] and it is limited to rather small bubbles (of the order of the Kolmogorov scale) and to potential flows. In Sec. III we employ the theory to find out at which Reynolds and Weber numbers the bubbles interact sufficiently strongly with the fluid to change significantly the stress tensor beyond simple viscosity renormalization. In Sec. IV we study the balance equation for momentum and energy in the turbulent boundary layer. This leads to the main section of this paper, Sec. V which presents the predictions of the theory regarding drag reduction by bubbles. The volume variations of the bubbles at sufficiently high Weber numbers are shown to be an important physical reason for the phenomenon. A summary and discussion are presented in Sec. VI.

II. AVERAGED EQUATIONS FOR BUBBLY FLOWS

A Newtonian fluid with density ρ is laden with bubbles of density ρ_B, and radius R which is much smaller than the outer scale of turbulence $\mathcal{L}$. The volume fraction of bubbles C is taken sufficiently small such that the direct interactions between bubbles can be neglected. In writing the governing equations for the bubbly flow we will assume that the length scales of interest are larger than the bubble radius. Later we will distinguish however, between the case of microbubbles whose radius is smaller than the Kolmogorov scale η and bubbles whose radius is of the order of η or slightly larger. For length scales larger than the bubbles one writes [15–17] the following:

(i) *The equation of motion for each bubble*

$$\rho_B C\dot{\mathbf{w}} = -C\nabla p + C\nabla\cdot\boldsymbol{\sigma} - C\boldsymbol{F},$$

$$\boldsymbol{F} \approx \frac{9\mu}{R^2}(\boldsymbol{U}-\boldsymbol{w}) + \frac{\rho}{2}\left(\frac{D\boldsymbol{U}}{Dt} - \dot{\boldsymbol{w}}\right), \quad (1)$$

where μ is the dynamical viscosity of the neat fluid. In this equation the force acting on the bubble is only approximate, since we neglect gravity, the lift force and the added mass force due to changes in the bubbles volume. We include only the viscous force and the add-mass force due to bubble acceleration, and we will show that this is sufficient for enhancing the drag reduction by the bubble dynamics. It can be argued that adding the other forces does not change things qualitatively.

(ii) *The equation of motion for the carrier fluid*

$$\rho(1-C)\frac{\partial\boldsymbol{U}}{\partial t} = -(1-C)\nabla p + (1-C)\nabla\cdot\boldsymbol{\sigma} + C\boldsymbol{F} + C\nabla\cdot\boldsymbol{\tau}. \quad (2)$$

(iii) *The continuity equation*

$$\frac{\partial(1-C)}{\partial t} + \nabla\cdot(1-C)\boldsymbol{U} = 0. \quad (3)$$

In these equations, $\boldsymbol{U}$ and $\boldsymbol{w}$ are the velocity of the carrier fluid and of the bubble, respectively, and

$$\sigma_{ij} \equiv \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \quad (4)$$

$\boldsymbol{F}$ and $\boldsymbol{\tau}$ are the force and the stress caused by the disturbance of the flow due to the bubbles, the Lagrangian derivatives are defined by

$$\frac{Da}{Dt} = \frac{\partial a}{\partial t} + \boldsymbol{U}\cdot\nabla a, \quad (5)$$

and

$$\dot{a} = \frac{\partial a}{\partial t} + \boldsymbol{w}\cdot\nabla a. \quad (6)$$

As the density of the bubble is usually much smaller than the fluid, ρ_B is taken to be 0. Combining (1) and (2), we have

$$\rho(1-C)\frac{D\boldsymbol{U}}{Dt} = -\nabla p + \nabla\cdot\boldsymbol{\sigma} + C\nabla\cdot\boldsymbol{\tau}. \quad (7)$$

Note that the term containing $\boldsymbol{F}$ disappears in the last equation because of the cancellation of action and reaction forces.

The bubbles affect the flow in two ways: (a) changing the effective density of the fluid; (b) introducing an additional stress tensor $\boldsymbol{\tau}$ to the fluid velocity equation (7).

The expression used for $\boldsymbol{\tau}$ is extremely important for the discussion at hand. It is commonly accepted that the stress tensor is affected by three factors:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_\nu + \boldsymbol{\tau}_R + \boldsymbol{\tau}_S. \quad (8)$$

In this equation $\boldsymbol{\tau}_\nu$ is the viscous stress tensor, written as follows:

$$\boldsymbol{\tau}_{\nu,ij} = \frac{5\mu}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right). \quad (9)$$

For very small bubbles (microbubbles) of very small density this is the only significant contribution in Eq. (8). When this is the case the bubble contribution to the stress tensor can be combined with $\boldsymbol{\sigma}$ in Eq. (7), resulting in the effective viscosity given by

$$\mu_{\text{eff}} = (1 + \tfrac{5}{2}C)\mu. \quad (10)$$

This formula is known to be correct when the bubbles are surface contaminated. The study of drag reduction under this renormalization of the viscosity and the concentration was presented in Ref. [8], with the result that drag reduction can be obtained by putting the bubbles out of the viscous sublayer and not too far from the wall. The amount of drag reduction is however rather limited in such circumstances.

The other two contributions in Eq. (8) are the concern of the present paper. The component $\boldsymbol{\tau}_R$ is nonzero only when the bubble is not a Lagrangian particle, having a relative velocity $\boldsymbol{w}-\boldsymbol{U}$ with respect to the fluid; then the bubble radius is changing in time. Explicitly [15–17],

$$\boldsymbol{\tau}_R = -\rho\left[\dot{R}^2 + \frac{3}{20}(\boldsymbol{w}-\boldsymbol{U})\cdot(\boldsymbol{w}-\boldsymbol{U})\right]\boldsymbol{I} - \frac{\rho}{20}(\boldsymbol{w}-\boldsymbol{U})(\boldsymbol{w}-\boldsymbol{U}). \quad (11)$$

The last contribution $\boldsymbol{\tau}_S$ is sensitive to the change in pressure of the fluid due to the bubbles. It reads [15]

$$\boldsymbol{\tau}_S = -\frac{R}{C}\int (p-p^0)\boldsymbol{nn}\, dA. \quad (12)$$

Here p^0 is the pressure of the fluid without bubbles, $\boldsymbol{n}$ is the normal unit vector to the bubble surface, and dA is the area differential. The relation of this expression to the relative velocity and to the bubble dynamics calls for a calculation, which in general is rather difficult. Such a calculation was achieved explicitly only for potential flows, with the final result [15,16]

$$\boldsymbol{\tau}_S = \rho\left[\frac{2}{5}(\boldsymbol{w}-\boldsymbol{U})\cdot(\boldsymbol{w}-\boldsymbol{U}) - R\ddot{R} - \frac{3}{2}(\dot{R})^2\right]\boldsymbol{I} - \frac{9\rho}{20}(\boldsymbol{w}-\boldsymbol{U})(\boldsymbol{w}-\boldsymbol{U}). \quad (13)$$

III. RELATIVE IMPORTANCE OF THE STRESS CONTRIBUTIONS AS A FUNCTION OF THE REYNOLDS NUMBER

The relative importance of the three contributions $\boldsymbol{\tau}_\nu$, $\boldsymbol{\tau}_R$, and $\boldsymbol{\tau}_S$ depends on the Reynolds number and on $R/\mathcal{L}$. To study this question we represent Eq. (1) as follows:

$$\frac{\rho}{2}\left(\frac{D\boldsymbol{U}}{Dt} - \dot{\boldsymbol{w}}\right) = -\nabla p + \nabla\cdot\boldsymbol{\sigma} + \frac{9\mu}{R^2}(\boldsymbol{U}-\boldsymbol{w}). \quad (14)$$

Consider first the case of small bubble size, $R < \eta$, and small Reynolds numbers. In this case the viscous term on the RHS is dominant, and the difference between $\boldsymbol{U}$ and $\boldsymbol{w}$ cannot be large. The bubbles behave essentially as Lagrangian tracers. On the other hand, at high values of Re and for larger bubbles, $R \geqslant \eta$, the term ∇p should be re-interpreted on the scale of the bubble as

$$\nabla p \approx \frac{p(\boldsymbol{x}+\boldsymbol{R}) - p(\boldsymbol{x}-\boldsymbol{R})}{2R} = \rho \frac{U^2(\boldsymbol{x}+\boldsymbol{R}) - U^2(\boldsymbol{x}-\boldsymbol{R})}{4R}, \tag{15}$$

where $\boldsymbol{x}$ is the location of the bubble. The second line in Eq. (15) follows from Bernouli's equation $p+\rho U^2/2 \approx \text{Const}$ (neglecting the acceleration due to gravity). When the size of the bubble becomes of the order of the Kolmogorov scale or larger, we have

$$U^2(\boldsymbol{x}+\boldsymbol{R}) - U^2(\boldsymbol{x}-\boldsymbol{R}) \sim 2U(\boldsymbol{x})(\epsilon R)^{1/3} \sim 2U(\boldsymbol{x})U_{\text{rms}}\left(\frac{R}{\mathcal{L}}\right)^{1/3}, \tag{16}$$

where U_{rms} is the rms of the turbulent velocity. At this point we can ask what is the value of the Reynolds number for which the viscous term is no longer dominant, allowing for significant fluctuations in $\boldsymbol{U}-\boldsymbol{w}$. This happens when the terms in Eq. (14) are comparable, i.e., when

$$|\boldsymbol{U}-\boldsymbol{w}| \sim \frac{U(\boldsymbol{x})U_{\text{rms}}}{R}\left(\frac{R}{\mathcal{L}}\right)^{1/3}\frac{R^2}{9\nu} \sim \frac{U_{\text{rms}}\text{Re}}{9}\left(\frac{R}{\mathcal{L}}\right)^{4/3}. \tag{17}$$

This equation contains an important prediction for experiments. It means that the fluctuations in the relative velocity of the bubble with respect to fluid is of the order of the outer fluid velocity when Re is larger than $(R/\mathcal{L})^{4/3}$. In most experiments, $R/\mathcal{L} \sim O(10^{-3})$ and it is therefore sufficient to reach $\text{Re} \sim \text{O}(10^5)$ for $|\boldsymbol{U}-\boldsymbol{w}|$ to be of the order of U_{rms}. Note that this is precisely the result of the experiment in Ref. [12].

This discussion has consequences for the bubble dynamics and compressibility. At small Re, $\boldsymbol{w}-\boldsymbol{U}$ is small and $\boldsymbol{\tau} \approx \boldsymbol{\tau}_\nu$. Then the equation of the mixture becomes

$$\rho_{\text{eff}}\frac{D\boldsymbol{U}}{Dt} = -\boldsymbol{\nabla} p + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_{\text{eff}} \tag{18}$$

with

$$\rho_{\text{eff}} = \rho(1-C), \tag{19}$$

$$\boldsymbol{\sigma}_{\text{eff}} = \boldsymbol{\sigma}\left(1+\tfrac{5}{2}C\right), \tag{20}$$

meaning that only the effective density and viscosity are changed, as is usually assumed in numerical simulations of "point" bubbles [12,13]. On the other hand, when Re is large $|\boldsymbol{w}-\boldsymbol{U}|$ is comparable to U_{rms}. This will affect the stress tensor on scales larger than the bubble size via $\boldsymbol{\tau}_{\text{R}}$ and $\boldsymbol{\tau}_{\text{S}}$. Furthermore,

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho}\left(p_{\text{B}} - \frac{2\gamma}{R} - p\right) + \frac{1}{4}(\boldsymbol{w}-\boldsymbol{U})\cdot(\boldsymbol{w}-\boldsymbol{U}) - 4\mu\frac{\dot{R}}{R}, \tag{21}$$

where γ is the surface tension. This equation tells us that the volume change of the bubbles is excited by the relative velocity $\boldsymbol{w}-\boldsymbol{U}$. When $\boldsymbol{w}-\boldsymbol{U}=0$, then

$$p_{\text{B}} = \frac{2\gamma}{R} + p, \tag{22}$$

and so R is a constant. Similarly, $\dot{R}$ is small if $\boldsymbol{w}-\boldsymbol{U}$ is small. The strength of the volume variations can be characterized by the Weber number

$$\text{We} \equiv \frac{\rho|\boldsymbol{w}-\boldsymbol{U}|^2 R}{\gamma}. \tag{23}$$

As a summary, the additional stress tensor $\boldsymbol{\tau}$ in the basic Eq. (7) due to the presence of bubble is a sum of three contributions, $\boldsymbol{\tau}_\nu$, $\boldsymbol{\tau}_{\text{R}}$, and $\boldsymbol{\tau}_{\text{R}}$, see Eq. (8). By using Eqs. (9), (11), and (13), we have

$$\begin{aligned}\boldsymbol{\tau} = \rho\{[\tfrac{1}{4}(\boldsymbol{w}-\boldsymbol{U})\cdot(\boldsymbol{w}-\boldsymbol{U}) - R\ddot{R} - \tfrac{5}{2}\dot{R}^2]\boldsymbol{I} \\ - \tfrac{1}{2}(\boldsymbol{w}-\boldsymbol{U})(\boldsymbol{w}-\boldsymbol{U}) + \tfrac{5}{2}\mu\boldsymbol{S}\},\end{aligned} \tag{24}$$

where the tensor $\boldsymbol{S}=\boldsymbol{\nabla}\boldsymbol{U}+\boldsymbol{\nabla}\boldsymbol{U}^T$. The relative importance of the various terms in $\boldsymbol{\tau}$ depends on the values of Re and We. If We is sufficiently large, there will be a large change in the diagonal part of $\boldsymbol{\tau}_{\text{S}}$. In the following section we show that this can be crucial for drag reduction.

IV. BALANCE EQUATIONS IN THE TURBULENT BOUNDARY LAYER

At this point we apply the formalism detailed above to the question of drag reduction by bubbles in a stationary turbulent boundary layer with plain geometry. This can be a pressure driven turbulent channel flow or a plain Couette flow, which is close to the circular Couette flow realized in Ref. [12]. Let the smallest geometric scale be $2L$ (for example the channel height in a channel flow), the unit vector in the streamwise and spanwise directions be $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$, respectively, and the distance to the nearest wall be $y \ll L$. The velocity $\boldsymbol{U}(\boldsymbol{r},\boldsymbol{t})$ has only one mean component, denoted by $\boldsymbol{V}=\hat{\mathbf{x}}V$, which depends only on y: $V=V(y)$. Denoting turbulent velocity fluctuations (with zero mean) by $\boldsymbol{u}(\boldsymbol{r},t)$ we have the Reynolds decomposition of the velocity field to its mean and fluctuating part,

$$\boldsymbol{U}(\boldsymbol{r},t) = V(y)\hat{\mathbf{x}} + \boldsymbol{u}(\boldsymbol{r},t). \tag{25}$$

Long time averages are denoted by $\langle\cdots\rangle$. Having dynamical equations (7) and (24), we can consider the effect of he bubbles on the statistics of turbulent channel flow. For this goal we shall use a simple stress model of planar turbulent flow. A similar model was successfully used in the context of drag reduction by polymeric additives [18]. This model is based on the balance equations of mechanical momentum, which we consider in the next Sec. IV A and the balance of the turbulent kinetic energy, discussed in Sec. IV B. The variables that enter the model are the mean shear

$$S \equiv dV/dy, \tag{26a}$$

the turbulent kinetic energy density

$$K \equiv \tfrac{1}{2}\rho(1-C)\langle|\boldsymbol{u}|^2\rangle, \tag{26b}$$

and the Reynolds stress

$$W_{xy} \equiv -\rho(1-C)\langle u_x u_y\rangle. \tag{26c}$$

A. Momentum balance

From Eq. (7) we derive the exact equation for the momentum balance by averaging and integrating in the usual way, and find for $y \ll L$,

$$P = \mu S + W_{xy} + C\langle\tau_{xy}\rangle. \tag{27}$$

Here P is the momentum flux toward the wall. In a channel flow $P=p'L$, where $p' \equiv -\partial p/\partial x$ is the (constant) mean pressure gradient. In a plain Couette flow P is another constant which is determined by the velocity difference between the two walls. For $C=0$ Eq. (27) is the usual equation satisfied by Newtonian fluids.

To expose the consequences of the bubbles we notice that the diagonal part of the bubble stress tensor $\boldsymbol{\tau}$ [the first line in the RHS of Eq. (24)] does not contribute to Eq. (27). The xy component of the off-diagonal part of $\boldsymbol{\tau}$ is given by the second line in Eq. (24). We define the dimensionless ratio

$$\alpha \equiv \frac{\langle(w_x - U_x)(w_y - U_y)\rangle}{2\langle u_x u_y\rangle}. \tag{28}$$

For later purposes it is important to assess the size and sign of α.

For small values of Re, α is small according to Eq. (18). On the other hand, it was argued in [19,20] that for large Re the fluctuating part of $\boldsymbol{w}$ is closely related to the fluctuating part of $\boldsymbol{u}$. The relation is

$$\boldsymbol{w} - \boldsymbol{U} \approx 2\boldsymbol{u}. \tag{29}$$

If we accept this argument verbatim this would imply that $\alpha \approx 2$ and is positive definite, as we indeed assume below. With this definition we can simplify the appearance of Eq. (27) to

$$P = \mu_{\text{eff}} S + \frac{1 + C(\alpha - 1)}{1 - C} W_{xy}, \tag{30}$$

with μ_{eff} defined by Eq. (10). Below we consider the high Re limit, and accordingly can neglect the first term on the RHS.

B. Energy balance

Next, we consider the balance of turbulent energy in the logarithmic layer. In this region, the production and dissipation of turbulent kinetic energy is almost balanced. The production can be calculated exactly, $W_{xy}S$. The dissipation of the turbulent energy is modeled by the energy flux which is the kinetic energy $K(y)$ divided by the typical eddy turn over time at a distance y from the wall, which is $\sqrt{\rho(1-C)}y/b\sqrt{K}$, where b is a dimensionless number of the order of unity. Thus the flux is written as $bK^{3/2}/y\sqrt{\rho(1-C)}$. The extra dissipation due to the bubble is $C\langle\tau_{ij}s_{ij}\rangle$, where $s_{ij} \equiv \partial u_i/\partial x_j$. In summary, the turbulent energy balance equation is then written as follows:

$$\frac{bK^{3/2}}{\sqrt{\rho(1-C)}y} + C\langle\tau_{ij}s_{ij}\rangle = W_{xy}S. \tag{31}$$

As usual, the energy and momentum balance equations do not close the problem, and we need an additional relation between the objects of the theory. For Newtonian fluids it is known that in the logarithmic layer W_{xy} and K are proportional to each other

$$W_{xy} = c_{\text{N}}^2 K, \quad c_{\text{N}} \approx 0.5. \tag{32a}$$

For the problem of drag reduction by polymers this ratio is also some constant $c_{\text{P}} \approx 0.25$ (in the maximum drag reduction regime). For the bubbly flow, we define c_{B} in the same manner

$$W_{xy} \equiv c_{\text{B}}^2 K. \tag{32b}$$

Clearly, $\lim_{C\to 0} c_{\text{B}} = c_{\text{N}}$ and for small C (noninteracting bubbles) $c_{\text{B}}^2 - c_{\text{N}}^2 \propto C$. It was reported in [21,22] that c_{B} is slightly smaller than its Newtonian counterpart; we therefore write

$$c_{\text{B}}^2 = c_{\text{N}}^2(1 - \beta C), \tag{32c}$$

with a positive coefficient β of the order of unity. We are not aware of direct measurements of this form in bubbly flows, but it appears natural to assume that the parameter β is y-independent in the turbulent logarithmic law region. We note that the Cauchy-Schwartz inequality can be used to prove that $W_{xy} \le K$, meaning that all the ratios $c_{\text{N,B,P}}^2 \le 1$.

V. DRAG REDUCTION IN BUBBLY FLOWS

In this section we argue that the bubble compressibility is crucial in enhancing the effect of drag reduction. This conclusion is in line with the experimental observation of Ref. [12], where bubbles and glass spheres were used under similar experimental conditions. Evidently, bubble deformations can lead to the compressibility of the bubbly mixture. Notwithstanding the difference between our notion of compressibility and that of [13], we note that in spirit this conclusion is in accord with the simulation of [13] where a strong correlation between compressibility and drag reduction were found.

To make the point clear we start with the analysis of the energy balance equation (31). The additional stress tensor τ_{ij} has a diagonal and an off-diagonal part. The off-diagonal part has a viscous part that is negligible for high Re. The other term can be evaluated using the estimate (29), leading to the contribution

$$\langle\tfrac{1}{2}(\boldsymbol{w} - \boldsymbol{U})(\boldsymbol{w} - \boldsymbol{U}):\boldsymbol{\nabla u}\rangle \approx 2\langle\boldsymbol{uu}:\boldsymbol{\nabla u}\rangle. \tag{33}$$

The expression on the RHS is nothing but the spatial turbulent energy flux which is known to be very small in the logarithmic layer compared to the production term on the RHS of Eq. (31). We will therefore neglect the off-diagonal part of the stress tensor in the energy equation. The analysis of the diagonal part of the stress depends on the issue of bubble compressibility and we therefore discuss separately oscillating bubbles and rigid spheres.

A. Drag reduction with rigid spheres

Consider first situations in which $\dot{R}=0$. This is the case for bubbles at small We, or when the bubbles are replaced by some particles which are less dense than neat fluid [12]. When the volume of the bubbles is fixed, the incompressibility condition for the Newtonian fluid is unchanged, and $s_{ii}=0$. The diagonal part of $\boldsymbol{\tau}$, due to the incompressibility condition $s_{ii}=0$, has no contribution to $\langle \tau_{ij} s_{ij} \rangle$. The energy balance equation is then unchanged compared to the Newtonian fluid. The momentum balance equation is nevertheless affected by the bubbles. Putting (32) into (31), we have

$$W_{xy} = \rho(1-C)\frac{S^2 y^2 c_{\mathrm{B}}^6}{b^2}. \tag{34}$$

To assess the amount of drag reduction we will consider an experiment [14] in which the velocity profile (and thus S) is maintained fixed. Drag reduction is then measured by the reduction in the momentum flux P. We then have

$$P = \frac{\rho(1-C+\alpha C)c_{\mathrm{B}}^6}{\kappa^2 b^2}, \tag{35}$$

where κ is the von Karman constant. If there are no bubbles ($C=0$), the Newtonian momentum flux P_{N} reads

$$P_{\mathrm{N}} = \frac{\rho c_{\mathrm{N}}^6}{\kappa^2 b^2}. \tag{36}$$

The percentage of drag reduction can be defined as

$$\%\mathrm{DR} = \frac{P_{\mathrm{N}} - P}{P_{\mathrm{N}}} = 1 - \frac{(1-C+\alpha C)c_{\mathrm{B}}^6}{c_{\mathrm{N}}^6} \approx (1-\alpha+3\beta)C. \tag{37}$$

Here we assumed that $\beta \ll 1$. At small Re, $\alpha=0$ and the amount of drag reduction increases linearly with C. If Re is very large we expect $\alpha \approx 2$, and then the drag is *enhanced*. This result is in pleasing agreement with the experimental data in [12]. Indeed, the addition of glass beads with density less than water caused drag *reduction* when Re is small, whereas at $\mathrm{Re} \sim (10^6)$, the drag was slightly *enhanced*.

B. Drag reduction with flexible bubbles

If the value of We is sufficiently large such that $\dot{R} \neq 0$, the velocity field is no longer divergenceless. To see how this affects the energy equation we consider a single bubble with volume $\mathcal{V}$. From the continuity equation

$$\int \boldsymbol{u} \cdot d\boldsymbol{A} = \dot{\mathcal{V}}. \tag{38}$$

If we assumed that the bubble is small enough such that the velocity field does not change much on the scale of R, then we have

$$\nabla \cdot \boldsymbol{u} \approx \frac{\dot{\mathcal{V}}}{\mathcal{V}} = 3\frac{\dot{R}}{R}. \tag{39}$$

Therefore, the last term in (21) can be approximated as

$$4\mu\frac{\dot{R}}{R} = \frac{4\mu}{3}\boldsymbol{\nabla} \cdot \boldsymbol{u}. \tag{40}$$

Next we substitute Eq. (21) into Eq. (24). For small amplitude volume variations we can neglect the terms proportional to $\dot{R}^2$ [23]. The expression for the stress tensor simplifies to

$$\tau_{ij} \approx \left[-p_{\mathrm{B}} + \frac{2\gamma}{R} + p + 4\rho\mu\frac{\dot{R}}{R}\right]\delta_{ij} - \frac{\rho}{2}(w_i - U_i)(w_j - U_j). \tag{41}$$

For large We, the term $\rho(w_i - U_i)(w_j - U_j)/2$ becomes larger than the terms $p_{\mathrm{B}} - 2\gamma/R + p$. Using Eq. (39)

$$\tau_{ij} \approx \rho\left[\frac{4}{3}\mu s_{ij}\delta_{ij} + \frac{1}{2}(w_i - U_i)(w_j - U_j)\right] \tag{42}$$

The extra turbulent dissipation due to the bubble is $\langle \tau_{ij} s_{ij} \rangle$. In light of the smallness of the term in Eq. (33) we find

$$\langle \tau_{ij} s_{ij} \rangle = \langle \tfrac{4}{3}\mu s_{ii}^2 \rangle. \tag{43}$$

The term $(4/3)\mu s_{ii}^2$ is of the same form as the usual dissipation term $\mu s_{ij} s_{ij}$ and therefore we write this as follows:

$$\langle \tfrac{4}{3}\mu s_{ii}^2 \rangle = A\frac{\rho|\boldsymbol{u}|^{3/2}}{y} = A\frac{K^{3/2}}{\sqrt{\rho}(1-C)^{3/2} y}, \tag{44}$$

where A is an empirical constant. Finally, the energy equation becomes

$$\frac{b(1-C) + AC}{\sqrt{\rho}(1-C)^{3/2}}\frac{K^{3/2}}{y} = W_{xy} S. \tag{45}$$

As before, we specialize the situation to an experiment in which S is constant, and compute the momentum flux

$$P = \frac{\rho(1-C)^2(1-C+\alpha C)}{(1-C+\frac{A}{b}C)^2}\frac{c_{\mathrm{B}}^6}{\kappa^2 b^2}. \tag{46}$$

The degree of drag reduction is then

$$\%\mathrm{DR} = 1 - \frac{(1-C)^2(1-C+\alpha C)}{(1-C+\frac{A}{b}C)^2}\left(\frac{c_{\mathrm{B}}}{c_{\mathrm{N}}}\right)^6 \approx \left(1-\alpha+\frac{2A}{b}+3\beta\right)C. \tag{47}$$

Note that A is an unknown parameter that should depend on We, and so its value is different in different experiments. The percentage of drag reduction for various values of A are shown in Fig. 1, where we chose $\alpha=2$ and for simplicity we estimate $\beta=0$. One sees that for $\alpha=2$ and $A=0$ (where the latter is associated with rigid bubbles), we only find drag enhancement. For small value of A, or small amplitudes of volume variations, small concentrations of bubbles lead (for $\alpha=2$) to drag enhancement, but upon increasing the concentration we find modest drag reduction. Larger values of A lead to considerably large degrees of drag reduction. For $A=0.15$, the result agrees reasonably with Legner's model which predicts $\%\mathrm{DR}\approx 1\text{–}5(1-C)^2/4$ [11]. Note that according to Legner, there should be considerable drag enhancement when $C=0$. This is of course a nonsensical result that is absent in our theory. For $A=0.8$, $\%\mathrm{DR}\approx 4C$ for small C.This is the best fit to the experimental results which are reported in [14].

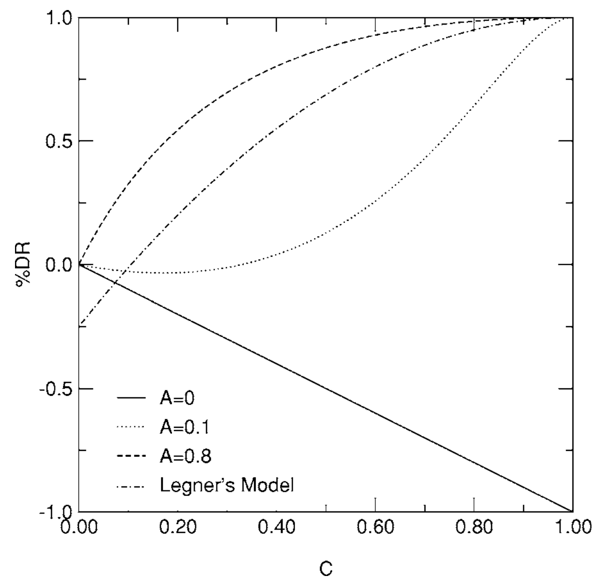

FIG. 1. Predicted values of drag reduction with $\alpha=2$ and different values of A. In a dashed line we reproduce the predictions of Legner's model which suffer from an unphysical drag enhancement at $C=0$. For $A=0$ (rigid spheres) we find only increasing drag enhancement as a function of C. For small values of A we have first a slight drag enhancement, and then modest drag reduction. For large values of A, associated with strong bubble volume variations, we find significant values of drag reduction.

VI. SUMMARY AND DISCUSSION

The main conclusion of this study is that bubble volume variations can contribute decisively to drag reduction by bubbles in turbulent flows. In agreement with the experimental findings of [12], we find that rigid bubbles tend to drag enhance, and the introduction of volume variations whose amplitude is measured by the parameter A (Fig. 1) increases the efficacy of drag reduction.

It is also important to recognize that bubble volume variations go hand in hand with the compressibility $\boldsymbol{\nabla}\cdot\boldsymbol{u}\neq 0$. In this sense we are in agreement with the proposition of [13] that drag reduction by bubbles is caused by the compressibility. There is a difference, however, in [13] the flow is free (having only one wall) whereas in our case we have a channel in mind. The mechanism of [13] cannot appear in our case. On the other hand [13] does not allow for bubble compressibility. The bottom line is that in both cases the bubble dynamics leads to the existence of compressibility, and the latter contributes to the drag reduction.

One drawback of the present study is that the bubble concentration is taken uniform in the flow. In reality a profile of bubble concentration may lead to even stronger drag reduction if placed correctly with respect to the wall. A consistent study of this possibility calls for the consideration of buoyancy and the self-consistent solution of the bubble concentration profile. Such an effort is beyond the scope of this paper and must await future progress.

Finally, it should be noted that we neglected the effects of viscosity in Eqs. (30) and (31) as we assumed the value of Re to be large. For moderately large Re, one can take the viscosity effects into account as suggested in [18].

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